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SO(3,1)-Valued Yang–Mills Fields

A. Reşit Dündarer[1]

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SO(3,1)-valued Yang–Mills fields are constructed on the four-dimensional manifold $M_4 = M_2 \times S_2$, where M_2 is a semiinfinite strip. It is shown that these fields have action proportional to the winding number of S_2 and the width of the strip and satisfy a self-duality relation of the form $*F = -i\gamma_5 F$. The Einstein tensor for the metric considered is found to be $G_{\mu\nu} = 3g_{\mu\nu}$.

In this article, I revise a previous work [1] where *SO*(3,1) Yang–Mills fields were constructed on a four-dimensional manifold $M_4 = M_2 \times S_2$ and the action was shown to be proportional to n, the winding number of S_2, and the width of the semiinfinite strip M_2. The primary field vector w, however, there was timelike, and the resulting fields are therefore for tachion fields. In this work, I take a spacelike vector w and study the resulting fields accordingly. The results, however, turn out to be very similar.

First I consider *SO*(3,1)-valued Yang–Mills fields in general. To this end, consider a four-vector w^μ, $\mu = 0, 1, 2, 3$. With the flat metric $\eta_{\mu\nu} = diag(+, -, -, -)$, denote

$$w^2 = (w^0)^2 - \vec{w} \cdot \vec{w} \tag{1}$$

As the line element, take

$$ds^2 = \frac{4dw^\mu \, dw^\nu \, \eta_{\mu\nu}}{(1 + w^2)^2} \tag{2}$$

so that we are considering a conformally flat metric. If we further take $w^\mu = w^\mu(x^\nu)$, we have

$$ds^2 = g_{\alpha\beta} \, dx^\alpha \, dx^\beta \tag{3}$$

[1] Physics Department, Middle East Technical University, Ankara 06531, Turkey.

where

$$g_{\alpha\beta} = \frac{4\eta_{\mu\nu}}{(1 + w^2)^2} \frac{\partial w^\mu}{\partial x^\alpha} \frac{\partial w^\nu}{\partial x^\beta} \tag{4}$$

The inverse metric elements are then

$$g^{\beta\gamma} = \frac{1}{4} (1 + w^2)^2 \eta^{\rho\sigma} \frac{\partial x^\beta}{\partial w^\rho} \frac{\partial x^\gamma}{\partial w^\sigma} \tag{5}$$

which satisfy $g_{\alpha\beta}g^{\beta\gamma} = \delta_\alpha^\gamma$. Now define the Dirac algebra-valued vector

$$w = w^\mu \gamma_\mu = w^0\gamma^0 - \vec{w} \cdot \vec{\gamma} \tag{6}$$

where γ^μ are the Dirac matrices

$$\gamma^0 = \gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = -\gamma_i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{7}$$

They satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} I \tag{8}$$

$$[\gamma_\mu, \gamma_\nu] = -2i\sigma_{\mu\nu} \tag{9}$$

where $\sigma_{\mu\nu}$ are the generators of $SO(3,1)$.

If we take as the gauge potential

$$A = \frac{[w, dw]}{2(1 + w^2)} \tag{10}$$

$$= \frac{-i\sigma_{\mu\nu}}{1 + w^2} w^\mu \, dw^\nu \tag{11}$$

from this we obtain the Yang–Mills field

$$F = dA + A \wedge A \tag{12}$$

$$= \frac{dw \wedge dw}{(1 + w^2)^2} \tag{13}$$

$$= \frac{-i\sigma_{\mu\nu}}{(1 + w^2)^2} dw^\mu \wedge dw^\nu \tag{14}$$

Using the property

$$\tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta} = -i\gamma_5\sigma^{\mu\nu} \tag{15}$$

we find that the duality relation

$$*F = -i\gamma_5 F \tag{16}$$

is satisfied by these fields. Here $\varepsilon^{0123} = -\varepsilon_{0123} = 1$ and

$$\gamma^5 = \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{17}$$

In the x space, we have

$$F = \tfrac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta \tag{18}$$

where

$$F_{\alpha\beta} = \frac{-2i\sigma_{\mu\nu}}{(1 + w^2)^2} \frac{\partial w^\mu}{\partial x^\alpha} \frac{\partial w^\nu}{\partial x^\beta} \tag{19}$$

Using the relations

$$[\sigma_{\mu\nu}, \sigma_{\alpha\beta}] = 2i(\eta_{\mu[\alpha}\sigma_{\beta]\nu} - \eta_{\nu[\alpha}\sigma_{\beta]\mu}) \tag{20}$$

$$\{\sigma_{\mu\nu}, \sigma_{\alpha\beta}\} = 2(\eta_{\mu[\alpha}\eta_{\beta]\nu} I + i\varepsilon_{\mu\nu\alpha\beta}\gamma_5) \tag{21}$$

we find

$$[F_{\mu\nu}, F_{\alpha\beta}] = g_{\mu[\alpha}F_{\beta]\nu} - g_{\nu[\alpha}F_{\beta]\mu} \tag{22}$$

$$\{F_{\mu\nu}, F_{\alpha\beta}\} = -\tfrac{1}{2}(g_{\mu[\alpha}g_{\beta]\nu} I + i\sqrt{-g}\,\varepsilon_{\mu\nu\alpha\beta}\gamma_5) \tag{23}$$

so that $F_{\mu\nu}$ are local representations of $SO(3,1)$ in curved space with metric $g_{\alpha\beta}$. We further find

$$F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} = -12I \tag{24}$$

so that the Yang–Mills action is

$$I_{\rm YM} = \frac{1}{2}\int Tr(F \wedge *F) \tag{25}$$

$$= -\frac{1}{4}\int \sqrt{-g}\; Tr(F_{\mu\nu}\, F^{\mu\nu})\, d^4x \tag{26}$$

$$= 12\int \sqrt{-g}\; d^4x \tag{27}$$

Now make the following parametrization:

$$w^0 = \sigma \cosh \tau \tag{28}$$

$$w^1 = \frac{\bar{z}^n + z^n}{1 + (z\bar{z})^n}\, \sigma \sinh \tau \tag{29}$$

$$w^2 = \frac{i(\bar{z}^n - z^n)}{1 + (z\bar{z})^n} \sigma \sinh \tau \tag{30}$$

$$w^3 = \frac{1 - (z\bar{z})^n}{1 + (z\bar{z})^n} \sigma \sinh \tau \tag{31}$$

Here σ, τ are real parameters with $\sigma \in [0, \infty]$ and $\tau \in [0, \Lambda]$, and $z \in S_2 \sim CP_1$ is a complex parameter. Note that

$$\vec{w} \cdot \vec{w} = \sigma^2 \sinh^2\tau \tag{32}$$

$$w^2 = (w^0)^2 - \vec{w} \cdot \vec{w} = \sigma^2 \tag{33}$$

Thus w is a space-like four-vector. This parametrization has a small difference from the parametrization of ref. 1 that causes a major difference in result. The small difference is that cosh τ and sinh τ are merely replaced in the parametrization. The major difference is that in ref. 1, $w^2 = -R^2 < 0$, while here, $w^2 = \sigma^2 > 0$. Thus the treatment in ref. 1 is for timelike fields, while here the treatment is for spacelike fields. The definitions of the gauge potential A and the Yang–Mills field F in Eqs. (10) and (13) are also modified accordingly. Now with this parametrization the Dirac algebra-valued vector $w = w^\mu \gamma_\mu$ can be written as

$$w = \sigma\xi \tag{34}$$

where

$$\xi = \begin{pmatrix} I \cosh \tau & -\hat{w} \sinh \tau \\ \hat{w} \sinh \tau & -I \cosh \tau \end{pmatrix} \tag{35}$$

with

$$\hat{w} = \frac{\vec{w} \cdot \vec{\sigma}}{|\vec{w}|} = \frac{1}{1 + (z\bar{z})^n} \begin{pmatrix} 1 - (z\bar{z})^n & 2\bar{z}^n \\ 2z^n & (z\bar{z})^n - 1 \end{pmatrix} \tag{36}$$

We have

$$\hat{w}^2 = I, \qquad \xi^2 = I \tag{37}$$

Then

$$dw = \xi \, d\sigma + \sigma \, d\xi \tag{38}$$

$$[w, dw] = 2\sigma^2 \xi \, d\xi \tag{39}$$

We obtain

$$d\xi = \begin{pmatrix} I \sinh \tau & -\hat{w} \cosh \tau \\ \hat{w} \cosh \tau & -I \sinh \tau \end{pmatrix} d\tau + \sinh \tau \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} d\hat{w} \tag{40}$$

$$\xi\, d\xi = -\begin{pmatrix} 0 & \hat{w} \\ \hat{w} & 0 \end{pmatrix} d\tau - \sinh \tau \begin{pmatrix} \hat{w} \sinh \tau & I \cosh \tau \\ I \cosh \tau & \hat{w} \sinh \tau \end{pmatrix} d\hat{w} \tag{41}$$

Further, we obtain

$$d\hat{w} = \frac{-2n}{1 + (z\bar{z})^n} [l\bar{z}^{n-1}\, d\bar{z} + l^\dagger z^{n-1}\, dz] \tag{42}$$

where l is the matrix given by

$$l = \frac{1}{1 + (z\bar{z})^n} \begin{pmatrix} z^n & -1 \\ z^{2n} & -z^n \end{pmatrix} \tag{43}$$

and which satisfies the relations

$$l^2 = 0, \qquad \{l, l^\dagger\} = I, \qquad [l, l^\dagger] = \hat{w} \tag{44}$$

$$\hat{w} l = l, \qquad \hat{w} l^\dagger = -l^\dagger \tag{45}$$

From these we also obtain

$$\hat{w}\, d\hat{w} = \frac{-2n}{1 + (z\bar{z})^n} [l\bar{z}^{n-1} d\bar{z} - l^\dagger z^{n-1}\, dz] \tag{46}$$

Putting these into Eq. (41), we obtain

$$\begin{aligned} \xi\, d\xi = {} & -b' d\tau + \frac{2n \sinh \tau}{1 + (z\bar{z})^n} \\ & \times [(f \sinh \tau + f' \cosh \tau)\bar{z}^{n-1} d\bar{z} \\ & + (-f^\dagger \sinh \tau + f'^\dagger \cosh \tau)\, z^{n-1}\, dz] \end{aligned} \tag{47}$$

where

$$b = \begin{pmatrix} \hat{w} & 0 \\ 0 & \hat{w} \end{pmatrix}, \qquad b' = \gamma_5 b = \begin{pmatrix} 0 & \hat{w} \\ \hat{w} & 0 \end{pmatrix} \tag{48}$$

$$f = \begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix}, \qquad f' = \gamma_5 f = \begin{pmatrix} 0 & l \\ l & 0 \end{pmatrix} \tag{49}$$

and which satisfy the relations

$$f^2 = f'^2 = 0 \tag{50}$$

$$[f, f^\dagger] = [f', f'^\dagger] = b \tag{51}$$

$$[f, f'^\dagger] = [f', f^\dagger] = b' \tag{52}$$

$$\{f, f^\dagger\} = \{f', f'^\dagger\} = I \tag{53}$$

$$\{f, f'^\dagger\} = \{f', f^\dagger\} = \gamma_5 \tag{54}$$

$$bf = f, \qquad bf^\dagger = -f\dagger, \qquad f^\dagger b = f^\dagger, \qquad fb = -f \tag{55}$$

$$b'f = bf' = f', \qquad b'f^\dagger = bf'^\dagger = -f'^\dagger \tag{56}$$

$$b'f' = f, \qquad b'f'^\dagger = -f^\dagger \tag{57}$$

$$b^2 = b'^2 = I, \qquad bb' = b'b = \gamma_5 \tag{58}$$

Thus the six fields $\{b, b', f, f^\dagger, f', f'^\dagger\}$ form a representation of flat $SO(3, 1)$ at every point (z, n) of $CP_1 \times Z^+$. Further, $\{b, f, f^\dagger\}$ represent the $SU(2)$ subalgebra of $SO(3,1)$, with b representing the $U(1)$ subalgebra of $SU(2)$ and the set $\{b', f', f'^\dagger\}$ representing the coset $SO(3, 1)/SU(2)$. Then we can write the gauge potential

$$A = \frac{[w, dw]}{2(1 + w^2)} = \frac{\sigma^2}{1 + \sigma^2}\, \xi\, d\xi \tag{59}$$

as

$$\begin{aligned} A = {} & \frac{2n\sigma^2 \sinh \tau}{(1 + \sigma^2)[1 + (z\bar{z})^n]} \\ & \times [(f \sinh \tau + f' \cosh \tau)\bar{z}^{n-1}\, d\bar{z} + (-f^\dagger \sinh \tau + f'^\dagger \cosh \tau) z^{n-1}\, dz] \\ & - \frac{\sigma^2}{1 + \sigma^2}\, b' d\tau \end{aligned} \tag{60}$$

Next, for the Yang–Mills field F we have

$$F = \frac{dw \wedge dw}{(1 + w^2)^2} \tag{61}$$

$$= \frac{2\sigma}{(1 + \sigma^2)^2}\, d\sigma \wedge \xi\, d\xi + \frac{\sigma^2}{(1 + \sigma^2)^2}\, d\xi \wedge d\xi \tag{62}$$

so that using Eqs. (40) and (41), we find

$$F = \frac{-2\sigma}{(1+\sigma^2)^2}\, b'd\sigma \wedge d\tau + \frac{\sigma^2}{(1+\sigma^2)^2}\,\frac{4n^2(z\bar{z})^{n-1}\sinh^2\tau}{[1+(z\bar{z})^n]^2}\, b\, dz \wedge d\bar{z}$$
$$+ \frac{4n\sigma\sinh\tau}{(1+\sigma^2)^2[1+(z\bar{z})^n]}$$
$$\times [(f\sinh\tau + f'\cosh\tau)\bar{z}^{n-1}d\sigma \wedge d\bar{z}$$
$$+ (-f^\dagger \sinh\tau + f'^\dagger\cosh\tau)z^{n-1}\, d\sigma \wedge dz$$
$$+ \sigma(f\cosh\tau + f'\sinh\tau)\bar{z}^{n-1}\, d\tau \wedge d\bar{z}$$
$$+ \sigma(-f^\dagger\cosh\tau + f'^\dagger\sinh\tau)z^{n-1}\, d\tau \wedge dz] \tag{63}$$

Now, setting $z = \rho e^{i\theta}$, with $\rho \in [0, \infty]$ and $\theta \in [0, 2\pi]$, we find the line element given in Eq. (2) as

$$ds^2 = \frac{4}{(1+\sigma^2)^2}\left[d\sigma^2 - \sigma^2\, d\tau^2 - \frac{4n^2\rho^{2n-2}}{(1+\rho^{2n})^2}\,\sigma^2\sinh^2\tau\,(d\rho^2 + \rho^2 d\theta^2)\right] \tag{64}$$

so that we have

$$\sqrt{-g} = \frac{4^3\sigma^3}{(1+\sigma^2)^4}\,\frac{n^2\rho^{2n-1}}{(1+\rho^{2n})^2}\sinh^2\tau \tag{65}$$

Then the Yang–Mills action given in Eq. (27) is

$$I_{\text{YM}} = 3\cdot 4^4 \int_0^\infty \frac{\sigma^3\, d\sigma}{(1+\sigma^2)^4}\int_0^\Lambda \sinh^2\tau\, d\tau \int_0^\infty \frac{n^2\rho^{2n-1}\, d\rho}{(1+\rho^{2n})^2}\int_0^{2\pi} d\theta \tag{66}$$

$$= 3\cdot 4^4\, \frac{1}{12}\,\frac{1}{4}\,(\sinh 2\Lambda - 2\Lambda)\,\frac{n}{2}\cdot 2\pi \tag{67}$$

$$= 16n\pi(\sinh 2\Lambda - 2\Lambda) \tag{68}$$

At this point, note that the polynomial $P_n(z) = (z - z_1)(z - z_2)\ldots(z - z_n)$ is homotopically equivalent to z^n, so that we can replace z^n with $P_n(z)$ without altering the results.

Finally, studying the geometrical properties of this metric on a computer, we obtain the Ricci tensor as

$$R_{\mu\nu} = -3g_{\mu\nu} \tag{69}$$

so that the Ricci scalar is

$$R = -12 \tag{70}$$

and the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} \tag{71}$$

obtains the value

$$G_{\mu\nu} = 3g_{\mu\nu} \tag{72}$$

Before concluding this article, I want to remark on its generalizations to higher dimensions. In refs. 2 and 3, $SO(5,1)$ and $SO(9,1)$ Yang–Mills fields were considered on the manifolds $M_6 = M_2 \times S_4$ and $M_{10} = M_2 \times S_8$, where the S_4 and S_8 instantons were embedded. First, these fields must also be modified by replacing sinh τ and cosh τ and by redefining A and F in accordence with Eqs. (10) and (13) so that we can have a spacelike manifold. However, the fields that represent flat $SO(5,1)$ and $SO(9,1)$ will remain the same. Also, in ref. 3, I obtained fields that represent the gauge group $SU(3) \times SU(2) \times SU(2) \times U(1)$ and on the cosets obtained fields that resemble a family of quarks. I believe all three families of quarks and leptons will emerge when one considers the 26-dimensional manifold $M_{26} = M_2 \times S_8 \times S_8 \times S_8$. How this generalization will be made is an outstanding problem.

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